Chapter 7: Techniques of Integration

Section 7.7: Approximate Integration

Approximate Integration

There are certain (definite) integrals whose value can not be found exactly

$$\int_0^{\pi} \sin\left(x^2\right) dx \qquad \int_0^1 e^{x^2} dx \qquad \int_{-1}^1 \sqrt{1+x^3}\, dx$$

Approximate Integration

Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where $\quad \Delta x = \dfrac{b-a}{n}$

and $\quad \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$

Derive…

Approximate Integration

Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where $\Delta x = \dfrac{b-a}{n}$

and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$

<u>Ex 1</u>: Use the midpoint rule with $n = 4$ to estimate $\int_0^1 e^x dx$. Then find the error in your estimate

Approximate Integration

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.

Derive…

Approximate Integration

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.

Ex 2: Use the trapezoidal rule with $n = 5$ to estimate $\int_1^4 \sqrt{x}\,dx$ then find the error in your estimate.

Approximate Integration

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots \right.$$

$$\left. + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where n is even and $\Delta x = (b - a)/n$.

Derive…

Approximate Integration

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where n is even and $\Delta x = (b - a)/n$.

<u>Ex 3</u>: Use Simpson's rule with $n = 8$ to estimate $\int_0^{\pi} \sin(x)\,dx$ then find the error in your estimate

Approximate Integration

Errors

$$E_M = \int_a^b f(x)dx - M_n$$

$$E_T = \int_a^b f(x)dx - T_n$$

$$E_S = \int_a^b f(x)dx - S_n$$

Approximate Integration

Errors

3 **Error Bounds** Suppose $|f''(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leqslant \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leqslant \frac{K(b-a)^3}{24n^2}$$

4 **Error Bound for Simpson's Rule** Suppose that $|f^{(4)}(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \leqslant \frac{K(b-a)^5}{180n^4}$$

Approximate Integration

Ex 4:

a) Use T_8 to approximate $\int_0^1 \cos(x^2)\,dx$

b) Estimate the error in using T_8 to approximate $\int_0^1 \cos(x^2)\,dx$

c) How large does n have to be so that T_n approximates $\int_0^1 \cos(x^2)\,dx$ to within 0.0001?

Approximate Integration

Ex 5: How large should n be to guarantee that the Simpson's Rule approximation to $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.00001?